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A DECOUPLING INEQUALITY FOR MULTILINEAR FUNCTIONS OF STABLE VECTORS*

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Abstract. This note contains the proof of a decoupling inequality for multilinear functions of symmetric B-valued stable random vectors.

1. Introduction. Decoupling inequalities were recently introduced by McConnell and Taqqu [8] for the study of double integrals with respect to symmetric stable processes. Subsequently, a number of authors have studied both decoupling inequalities and their applications to multiple stochastic integration ([3]–[6], [9], [10]).

In the present note we prove a decoupling inequality for multilinear functions of symmetric B-valued stable random vectors. Although there is a partial overlap with decoupling inequalities proved by other authors, our result is more complete in the case of symmetric p-stable vectors, since it covers all powers $\|\cdot\|^q$ with $0 < q < p$. In addition, our method of proof is very simple.

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2. The decoupling inequality. Let B, V be separable Banach spaces. Let $d \in N$ and let M: $B^d \to V$ be a measurable symmetric multilinear map. Let X be a symmetric p-stable B-valued r.v. $(0 < p < 2)$ and let X_i, $i = 1, \ldots, d$, be independent copies of X. In what follows, it will be assumed that the following integrability condition is satisfied: for a fixed $q \in (0, p)$,

$$E\|\tilde{M}(X)\|^q < \infty, \tag{1}$$

where $\tilde{M}(x) = M(x, x, \ldots, x)$.

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THEOREM. *For every $p\in(0,2)$, $q\in(0,p)$, $d\in N$, there exist constants $c = c(p,q,d)$ and $C = C(p,q,d)$ such that, for any separable Banach spaces B, V, for any symmetric p-stable B-valued r.v. X and any measurable symmetric multilinear map $M\colon B^d\to V$ satisfying* (1), *the following inequalities hold*:

$$cE\|M(X_1,\ldots,X_d)\|^q \leqslant E\|\tilde{M}(X)\|^q \leqslant CE\|M(X_1,\ldots,X_d)\|^q.$$

Proof. We shall use the following notation:

$$x^k y^{d-k} = (\overbrace{x,\ldots,x}^{k},\overbrace{y,\ldots,y}^{d-k}).$$

For example, if $\pi\colon B^d\to B^d$ is a permutation of coordinates, then by the symmetry of M we have $M\big(\pi(x^k y^{d-k})\big) = M(x^k y^{d-k})$. Also, $M(x^d) = \tilde{M}(x)$.

(I) The left inequality follows from the general polarization identity (see e.g. [2], p. 80, and references therein)

$$(2)\qquad (2^d d!)\,M(x_1,\ldots,x_d) = \sum_{\varepsilon\in I^d}\varepsilon_1\varepsilon_2\ldots\varepsilon_d\,\tilde{M}\Big(\sum_{j=1}^{d}\varepsilon_j x_j\Big),$$

where $I = \{-1, 1\}$ and $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_d)$. In fact, since

$$\mathscr{L}\Big(d^{-1/p}\sum_{j=1}^{d}\varepsilon_j X_j\Big) = \mathscr{L}(X)\qquad\text{for every } \varepsilon\in I^d,$$

we have from (2) and the triangle inequality: for $q\geqslant 1$ (so in this case $p>1$),

$$\begin{aligned}\big(E\|M(X_1,\ldots,X_d)\|^q\big)^{1/q} &\leqslant (2^d d!)^{-1}\,2^d d^{d/p}\big(E\|\tilde{M}(X)\|^q\big)^{1/q}\\ &= (d^{d/p}/d!)\big(E\|\tilde{M}(X)\|^q\big)^{1/q}.\end{aligned}$$

This shows that the left inequality is valid in this case with $c(p,q,d) = (d!\,d^{-d/p})^q$.

For $0<q<1$, $E\|M(X_1,\ldots,X_d)\|^q \leqslant (2^d d!)^{-q}\,2^d d^{dq/p} E\|\tilde{M}(X)\|^q$, so in this case the left inequality holds with $c(p,q,d) = 2^{d(q-1)}(d!\,d^{-d/p})^q$.

(II) To prove the right inequality we proceed in two steps. The first step is to prove the following claim:

(3) the right inequality is true for $q\in(p/2,p)$.

To prove claim (3) we proceed by induction on d. If $d=1$, there is nothing to prove. Assume that $d>1$ and (3) is true for $1\leqslant n<d$. Let $\gamma = 2^{-1/p}$; then $\mathscr{L}\big(\gamma(X+Y)\big) = \mathscr{L}(X)$, where Y is an independent copy of X. If $q\geqslant 1$, then

$$\begin{aligned}E\|\tilde{M}(X)\|^q = E\big\|M\big((\gamma(X+Y))^d\big)\big\|^q &= \gamma^{dq}E\Big\|\sum_{n=0}^{d}\binom{d}{n}M(X^n Y^{d-n})\Big\|^q\\ &\leqslant \gamma^{dq}E\Big(\sum_{n=0}^{d}\binom{d}{n}\|M(X^n Y^{d-n})\|\Big)^q\end{aligned}$$

and, therefore,

(4) $$\left(E\|M(X^d)\|^q\right)^{1/q} \leqslant \gamma^d \sum_{n=0}^{d} \binom{d}{n} \left(E\|M(X^n Y^{d-n})\|^q\right)^{1/q}$$

$$= 2\gamma^d \left(E\|M(X^d)\|^q\right)^{1/q} + \gamma^d \sum_{n=1}^{d-1} \binom{d}{n} \left(E\|M(X^n Y^{d-n})\|^q\right)^{1/q}.$$

Observe that all terms on the right-hand side are finite; this follows from (1), (2) and the triangle inequality. Now, if $\mu = \mathscr{L}(X)$,

$$E\|M(X^n Y^{d-n})\|^q = \int d\mu(x) E\|M(x^n Y^{d-n})\|^q.$$

The inner expectation is finite for almost all x. Since $d-n \leqslant d-1$, by the inductive hypothesis, for almost all x,

$$E\|M(x^n Y^{d-n})\|^q \leqslant C(p, q, d-n) E\|M(x^n, X_{n+1}, \ldots, X_d)\|^q.$$

Thus

(5) $$E\|M(X^n Y^{d-n})\|^q \leqslant C(p, q, d-n) E\|M(X^n, X_{n+1}, \ldots, X_d)\|^q$$

with $X^n, X_{n+1}, \ldots, X_d$ independent copies of X. Next,

(6) $$E\|M(X^n, X_{n+1}, \ldots, X_d)\|^q$$

$$= \int d\mu^{d-n}(x_{n+1}, \ldots, x_d) E\|M(X^n, x_{n+1}, \ldots, x_d)\|^q.$$

Again, the inner expectation is finite for almost all $(x_{n+1}, \ldots, x_d)$. Since $n \leqslant d-1$, by the inductive hypothesis, for almost all $(x_{n+1}, \ldots, x_d)$,

(7) $$E\|M(X^n, x_{n+1}, \ldots, x_d)\|^q \leqslant C(p, q, n) E\|M(X_1, \ldots, X_n, x_{n+1}, \ldots, x_d)\|^q.$$

From (5)–(7) we get

(8) $$E\|M(X^n Y^{d-n})\|^q \leqslant C(p, q, d-n) C(p, q, n) E\|M(X_1, \ldots, X_d)\|^q.$$

From (4) and (8) we get

(9) $$(1-2\gamma^d)\left(E\|\tilde{M}(X)\|^q\right)^{1/q}$$

$$\leqslant \left[\gamma^d \sum_{n=1}^{d-1} \binom{d}{n} C(p, q, d-n)^{1/q} C(p, q, n)^{1/q}\right] \left(E\|M(X_1, \ldots, X_d)\|^q\right)^{1/q}.$$

Since $p < 2$ and $d \geqslant 2$, it follows that $2\gamma^d = 2^{1-(d/p)} < 1$ and, therefore,

$$E\|\tilde{M}(X)\|^q \leqslant (1-2\gamma^d)^{-q} D^q E\|M(X_1, \ldots, X_d)\|^q,$$

where

$$D = \gamma^d \sum_{n=1}^{d-1} \binom{d}{n} C(p, q, d-n)^{1/q} C(p, q, n)^{1/q}.$$

If $p/2 < q < 1$, using the elementary inequality

$$\Big(\sum_{i=1}^{m} a_i\Big)^q \leqslant \sum_{i=1}^{m} a_i^q \qquad (a_i \geqslant 0)$$

and proceeding in a similar way, we obtain in the inductive step

$$E\,\|\tilde{M}(X)\|^q \leqslant 2\gamma^{dq} E\,\|\tilde{M}(X)\|^q + \\ + \Big[\gamma^{dq} \sum_{n=1}^{d-1} \binom{d}{n}^q C(p,q,d-n)\,C(p,q,n)\Big] E\,\|M(X_1,\ldots,X_d)\|^q.$$

Since $d \geqslant 2$ and $q > p/2$, it follows that $2\gamma^{dq} = 2^{1-(dq/p)} \leqslant 2^{1-(2q/p)} < 1$, so in this case we have

$$E\,\|\tilde{M}(X)\|^q \leqslant (1-2\gamma^{dq})^{-1} D E\,\|M(X_1,\ldots,X_d)\|^q,$$

where $D = \gamma^{dq} \sum_{n=1}^{d-1} \binom{d}{n}^q C(p,q,d-n)\,C(p,q,n)$.

This proves claim (3).

(III) In order to complete the proof of the theorem we need the following

LEMMA. *Let $M\colon B^d \to V$ be a measurable symmetric multilinear map. Let X be a p-stable symmetric B-valued r.v. and let $X_1, \ldots, X_d$ be independent copies of X. Then:*

(a) *for every $q \in (0, p)$,*

$$E\,\|M(X_1,\ldots,X_d)\|^q < \infty;$$

(b) *for every $0 < q < r < p$ there exists a constant $A = A(p,q,r,d)$ (depending only on p, q, r, d) such that*

$$\big(E\,\|M(X_1,\ldots,X_d)\|^r\big)^{1/r} \leqslant A\big(E\,\|M(X_1,\ldots,X_d)\|^q\big)^{1/q}.$$

Proof. We first need to extend certain well-known results for stable B-valued r.v.'s to a more general situation(*). Since the arguments are slight modifications of standard ones in the B-valued case, we will merely sketch them. Let E be a real vector space and let $0 < \alpha \leqslant 1$ be fixed. Assume that $\varrho\colon E \to R^+$ is an α-homogeneous quasi-norm; that is, ϱ satisfies

(i) $\qquad \varrho(x+y) \leqslant \varrho(x)+\varrho(y) \quad$ for $x, y \in E$,

(ii) $\qquad \varrho(\lambda x) = |\lambda|^\alpha \varrho(x) \quad$ for $x \in E$, $\lambda \in E$,

and that E is a separable metric space with the metric $d(x,y) = \varrho(x-y)$. We set $\|x\|_\alpha = (\varrho(x))^{1/\alpha}$; then

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(*) We are indebted to B. Rajput for a question that led us to clarify this point.

$$\|x+y\|_\alpha \leqslant a(\|x\|_\alpha+\|y\|_\alpha) \quad \text{for } x, y\in E, \text{ where } a = 2^{(1/\alpha)-1},$$
$$\|\lambda x\|_\alpha = |\lambda|\,\|x\|_\alpha \quad \text{for } x\in E,\ \lambda\in \boldsymbol{R}.$$

If Y_j, $j = 1, \ldots, n$, are independent symmetric E-valued r.v.'s and $S_n = \sum Y_j$ $(j = 1, \ldots, n)$, then the following Lévy-type inequality is obtained by an obvious modification of the usual proof:

$$P\{\sup_{k\leqslant n}\|Y_k\|_\alpha > at\} \leqslant 2P\{\|S_n\|_\alpha > t\} \quad \text{for } t > 0. \tag{10}$$

If the Y_j's are independent copies of a symmetric p-stable r.v. Y, then by a standard argument we get from (10): for all $n\in N$, $t > 0$,

$$nP\{\|Y\|_\alpha > atn^{1/p}\} \leqslant -\log(1-2P\{\|Y\|_\alpha > t\}).$$

From this inequality it follows that

$$E\|Y\|_\alpha^r < \infty \quad \text{for } 0 < r < p, \tag{11}$$

$$(E\|Y\|_\alpha^r)^{1/r} \leqslant C(E\|Y\|_\alpha^q)^{1/q} \quad \text{for } 0 < q < r < p, \tag{12}$$

where the constant C depends only on p, q, r, α. Of course, (11) and (12) are well-known results if $\alpha = 1$ (see e.g. [1], Th. 3.2, and [7], Prop. 7.3.4).

We pass now to the proof of statements (a) and (b).

(a) We proceed by induction. Let $0 < q < p$. For $d = 1$ the assertion reduces to (11). Assume that the result is true for $n = d-1$. For each $x\in B$, let $\varphi(x)$ be the $(d-1)$-multilinear map on B^{d-1} defined by

$$\varphi(x)(x_1, \ldots, x_{d-1}) = M(x_1, \ldots, x_{d-1}, x).$$

By the inductive hypothesis, $\varphi(x)\in L^q(B^{d-1}, \mu^{d-1}; V)$. Moreover, the map $\varphi\colon B\to L^q(B^{d-1}, \mu^{d-1}; V)$ is a measurable linear map, and hence $Z = \varphi(X_d)$ is a symmetric p-stable $L^q(B^{d-1}, \mu^{d-1}; V)$-valued r.v. Then

$$\begin{aligned} E\|M(X_1, \ldots, X_d)\|^q &= \int d\mu(x)\int\|M(x_1, \ldots, x_{d-1}, x)\|^d\, d\mu(x_1)\ldots d\mu(x_{d-1}) \\ &= \int\|\varphi(x)\|_q^q\, d\mu(x) = E\|Z\|_q^q < \infty \end{aligned}$$

by (11), applied to $E = L^q(B^{d-1}, \mu^{d-1}; V)$ and

$$\varrho(f) = \begin{cases} \int\|f\|^q\, d\mu^{d-1} & \text{if } 0 < q < 1 \text{ (so } \alpha = q), \\ (\int\|f\|^q\, d\mu^{d-1})^{1/q} & \text{if } q \geqslant 1 \text{ (so } \alpha = 1). \end{cases}$$

(b) From (12) it follows that there exists a constant $A(p, q, r)$ such that, for every symmetric p-stable r.v. taking values in a vector space E with an α-homogeneous quasi-norm, where $\alpha = 1$ or $\min(1, q)$,

$$(E\|Y\|_\alpha^r)^{1/r} \leqslant A(p, q, r)(E\|Y\|_\alpha^q)^{1/q}. \tag{13}$$

We will prove the claimed statement with

$$A(p, q, r, d) = \big(A(p, q, r)\big)^d.$$

Again we proceed by induction. For $d = 1$ the statement reduces to (13). Assume that the assertion is true for $n = d-1$. Then, proceeding as before and using (13),

$$\begin{aligned} E\|M(X_1, \ldots, X_d)\|^r &= \int d\mu(x) E\|M(X_1, \ldots, X_{d-1}, x)\|^r \\ &\leqslant \big(A(p, q, r)\big)^{(d-1)r} \int \big(E\|M(X_1, \ldots, X_{d-1}, x)\|^q\big)^{r/q} d\mu(x) \\ &= \big(A(p, q, r)\big)^{(d-1)r} E\|Z\|_q^r \leqslant \big(A(p, q, r)\big)^{(d-1)r} \big(A(p, q, r)\big)^r \big(E\|Z\|_q^q\big)^{r/q} \\ &= \big(A(p, q, r)\big)^{dr} \big(E\|M(X_1, \ldots, X_d)\|^q\big)^{r/q}. \end{aligned}$$

This completes the proof of the lemma.

We can finally complete the proof of the theorem. If $0 < q \leqslant p/2$, choose $r \in (p/2, p)$. Then by Hölder's inequality, claim (3) and the Lemma,

$$\begin{aligned} \big(E\|\hat{\varphi}(X)\|^q\big)^{r/q} &\leqslant E\|\hat{\varphi}(X)\|^r \leqslant CE\|M(X_1, \ldots, X_d)\|^r \\ &\leqslant CA^r \big(E\|M(X_1, \ldots, X_d)\|^q\big)^{r/q}. \end{aligned}$$

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